Networks, Percolation, and Demand*

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Abstract

We propose a diffusion model for a new product percolating in a social network. Given that consumers differ in their reservation prices, a critical price exists that defines a phase transition from a *no-diffusion* to a *diffusion* regime. As consumer surplus is maximized just below a product's critical price, one can systematically compare the economic efficiency of network structures by investigating their critical price. Networks with low clustering turn out to be most efficient, because clustering leads to redundant information flows hampering effective product diffusion. We further show that the more equal a society, the less diffusion is hampered by clustering.

1 Introduction

The success of innovations largely depends on the diffusion process (Griliches, 1957; Mansfield, 1961; Bass, 1969; Davies, 1979). The question how innovations diffuse is among the

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core questions in economics, sociology, marketing, innovation studies, and even physics (Stoneman, 2001; Rogers, 2003; Valente, 2005; Vega-Redondo, 2007). The role of social networks in innovation diffusion has become a key question since innovations generally diffuse through patterns of social interactions (Banerjee et al., 1999). Hence, the structure of social networks in societies may bear important consequences for the rate of technological progress and economic growth (Fogli and Veldkamp, 2014). Our work builds on these questions by analysing the welfare effects of social networks in new product diffusion.

Though innovations have complex effects on consumers' welfare, the speed and extent of diffusion are arguably the two aspects of the diffusion process that are both universal and fundamental. If one assumes that any consumer adopting an innovation is improving its welfare, both a faster diffusion and a mode widespread diffusion automatically imply higher returns to society. The key question becomes how different network structures impact on the speed and extent of innovation diffusion.

Starting from the generic assumption that consumers differ in their reservation prices, we show that the network structures supporting the speed of diffusion are not necessarily those that lead to a greater extent of diffusion and *vice versa*. In particular, we show that small-world networks, which exhibit short path lengths and high clustering (Watts and Strogatz, 1998), lead to fast but often only limited diffusion compared to fully random networks, which exhibit short path lengths and low clustering. The reason for limited diffusion in small-world networks is the redundancy in information spreading due to clustering. Accordingly, consumer welfare is much lower in small-world networks than in random networks for a wide ranges of prices.

Our main contributions are threefold. First, we translate standard percolation theory as it has already been applied to innovation diffusion before (Solomon et al., 2000; Hohnisch et al., 2008; Cantono and Silverberg, 2009), into an explicit welfare-theoretical framework in which the inefficiency of networks can be expressed by the unfulfilled consumer surplus. We can then derive that network efficiency is equivalent to the critical price below which diffusion becomes complete. Second, we show that for a standard linear demand curve with uniformly distributed reservation prices, the inefficiency of social networks is a function of its clustering: the higher the degree of clustering in a network ("friends-of-friends-being-friends", the less efficient a social network in diffusing an innovation. This finding is important as it runs counter the common belief that small-world networks, which combine high clustering with short path lengths, are supportive of economic welfare [REF's??]. Third, the detrimental welfare effects of clustering, in turn, decline for lower levels variance in consumers' preference distribution. As consumers' preferences become more alike, so does the critical price below which full diffusion occurs, for different network topologies.

The remainder of the article is organised as follows. Section 2 studies the effects of percolation on innovation diffusion. Section 3 looks at percolation in small-world networks. Section 4 analyses the structural factors of diffusion. Section 6 addresses alternative demand curves, with a non-uniform distribution of consumers' reservation prices. Section 7 studies percolation in scale-free networks. Section 8 concludes.

2 Percolation and demand

Consider a new product and a network of N potential consumers, where i and j are neighbours if there is a link $\eta_{i,j}$ connecting them. Links are either existing ($\eta_{i,j} = 1$) or absent ($\eta_{i,j} = 0$), and do not depend on time. The diffusion process starts exogenously with a small number $n << N$ of initial adopters of the new product (seeds). Information about the innovative product is local: consumers who are not among the initial adopters, come to know about the new product only if a neighbour adopts.

The adoption decision is based on the product's price p, which is defined in the interval $[0, 1]$, and which is assigned before diffusion starts. Consumers' preferences are expressed by a reservation price $p_i \sim U[0, 1]$: only consumers with $p_i > p$ are *willing-to buy* (Fig. 1, central panel). A consumer who does not adopt does not pass on information to her neighbours, while a consumer who adopts passes on information to her neighbours. Hence, the network that truly matters for diffusion is the one that results from individual reservation prices. Drawing consumers reservation prices amounts to randomly switching 'off' nodes and their links (Fig. 1, right panel). The resulting network of active nodes is called the *operational network*. Diffusion will have a sizeable extent only if a large ('giant') connected component exists in the operational network and at least one seed is part of this giant component.

What is important to notice is that drawing reservation price has a highly non-linear effect on diffusion. In the example of Fig. 1 we have 30 consumers, and all but one happen

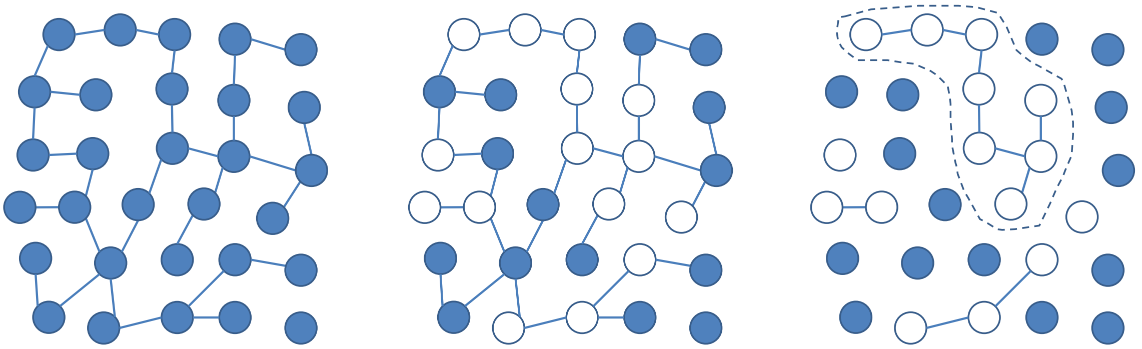


Figure 1: Percolation in a network of consumers considering whether to buy an innovative product with price $p \in [0,1]$. Left: original network. Centre: after drawing reservation prices $p_i \sim U[0,1]$, only consumers with $p_i > p$ are *willing-to-buy* (white nodes), while the others are not (filled nodes). Right: consumers that are not willing to buy are removed from the networks, and their links are removed as well. The resulting giant connected component is enlightened with the dashed line.

to be connected initially (Fig. 1, left panel). An innovation price $p = 0.5$ means that on average 50% of consumers are *willing-to buy*, that is 15 consumers. The *unwilling-to-buy* consumers are "removed", and the resulting network is made of a number of connected components, the largest being formed by only eight consumers. In the best case, when we have an initial adopter who belongs to this component, diffusion size will be eight, which is just over half the potential diffusion size of 15 *willing-to-buy* consumers.

If all consumers are directly linked in a fully connected network (meaning that all consumers have $N-1$ neighbours), all consumers are immediately informed once any seed adopts, and diffusion will always attain its maximum possible size. Assuming a uniform distribution of consumers' reservation prices, we obtain the standard downward-sloping linear demand curve. When p is the price of the innovative product, the probability that a consumer is *willing-to-buy* is $q = 1-p$, and the expected number of adopters is $N(1-p)$.

It is the local nature of information diffusion in social networks that makes the actual adoption curve to depart from the standard linear demand curve. In particular, a sparse network structure introduces a phase transition, leaving many potential adopters uninformed at high prices. We illustrate this point by simulating the percolation model for the case where 10,000 consumers form a Poisson random network (Erdös and Rényi, 1960). The left panel of Fig. 2 reports the final number of adopters for different values of the innovation price. The percolation process is initiated by 10 seeds ($n = 10$).[1] At low

[1]The exact diffusion size depends on the number of seeds, but the critical transition threshold does not. Simulations with 100 seeds yield the same patterns. Simulations with less than 10 seeds are less informative, since the variability of results is too large, with a standard deviation near 100%. The reason

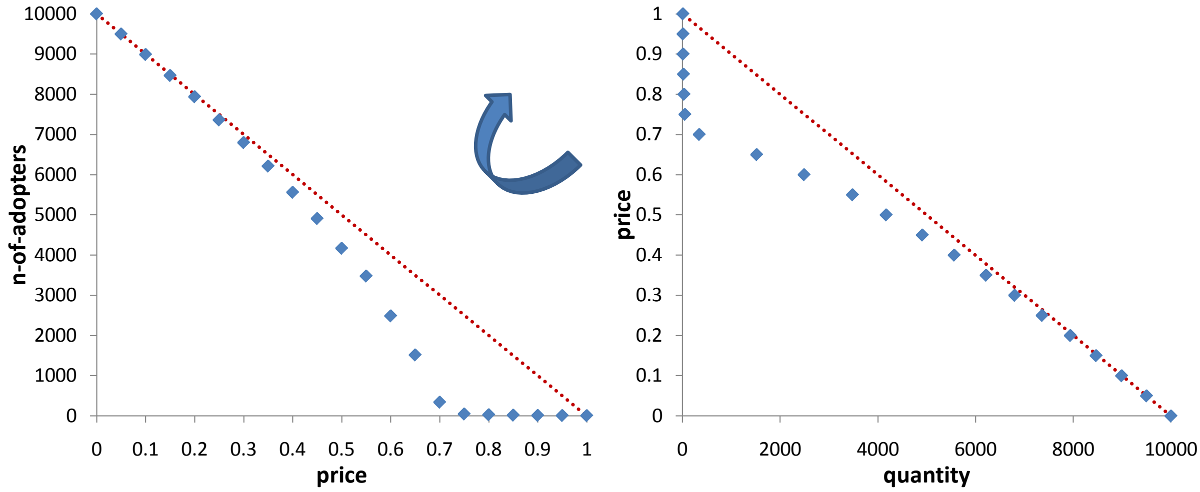


Figure 2: Percolation in a Poisson network with $N = 10000$ nodes, average connectivity 4, and 10 seeds. Left: number of adopters as a function of the products price. Left: adoption values reported as a demand curve. Values are averages over 50 simulation runs. The standard deviation is largest at the percolation threshold (25%).

prices, the number of adopters follows the linear demand curve reflecting the uniform distribution of reservation prices. This amounts to full diffusion of the innovation, meaning that all consumers with a reservation price equal or above the products price, actually adopt the product. However, already at values as low as $p = 0.4$ the diffusion size starts to be lower than the number of consumers willing to adopt, and drops down to almost zero above $p = 0.7$. In these scenarios diffusion is not full, because the information of the product's existence does not spread in the network. Put differently, the operational network is fragmented in many small components of consumers, without any links between these components. As we increase the price, an ever increasing share of consumers that would be *willing-to-buy* the innovation based on its price, do not actually buy it, because they do never get to know about its existence (Solomon et al., 2000).

We can distinguish between two different regimes: a *diffusion* regime, where information spreads throughout the whole operational network, and diffusion is full or almost-full. A *no-diffusion* regime, where information does not spread, and diffusion is very limited leaving many potential buyers uninformed. By lowering the price, the consumers' network undergoes what in physics is called a percolation *phase transition* (Stauffer and Aharony, 1994). A critical threshold value of the price, p_c, separates the two regimes (phases). This is the value that marks a fundamental change in the structure of the system, namely the value below which we see the appearance of a *giant connected component* of consumers

is that if no seeds fall in the giant connected component, no macroscopic diffusion takes place.

that are willing-to-buy (see Fig. 1). The size of the giant component is highly non-linear, with a sharp increase in size below the critical price p_c (the percolation threshold).

The percolation threshold is a mathematical property of a network, which can be computed, at least numerically. A powerful approach is based on *Generating Functions*, introduced by Newman et al. (2001) for the analysis of several characteristics of random networks. Callaway et al. (2000) apply this formalism to percolation. The case of a Poisson network allows to evaluate analytically the percolation threshold. In the price space dimension of innovation diffusion, and for a network of infinite size, the percolation threshold is given by

$$p_c = \frac{\langle k \rangle - 2}{\langle k \rangle - 1}, \tag{1}$$

where k refers to the connectivity of nodes. In the example of Fig. 2 the average connectivity is $\langle k \rangle = 4$, so the percolation threshold is $p_c \simeq 0.67$.

The effects of social networks can be expressed in welfare-theoretical terms. Network inefficiency stems from "lost" consumers, that is, consumers who would have been willing to buy, but who are not informed by any neighbour about the product. That is, for relatively expensive products, a social network may not be able to convey to all its members about the product's existence, and consequently not all consumers that would like to buy the product will buy it, because they do not get informed. The group of consumers who, at price p, are willing to buy but who do not adopt, we will refer to as lost consumers. By definition, a lost consumer has a reservation price in the range between 1 and p. Hence, given the uniform distribution of reservation prices, the expected loss in consumer surplus of a lost consumer amounts to $(1-p)/2$. Accordingly, in the no-diffusion regime, the welfare loss amounts to $N(1-p)$ lost consumers who miss out, on average, on $(1-p)/2$ surplus, amounting to

$$\frac{1}{2}N(1-p)^2 \quad \text{for} \quad p > p_c. \tag{2}$$

Hence, for a given society with N members, the welfare loss due to network inefficiency is solely a function of the price. This implies that, in the following analysis of network inefficiencies, we can proceed by solely focusing on the critical price that separates the diffusion regime from the no-diffusion regime. That is, we can express the inefficiency of different network topologies by the critical price below which full diffusion occurs. The lower this critical price, the less efficient is the network in question.

3 Innovation diffusion in a small-world network

One of the most popular models of social networks is the small-world model introduced by Watts and Strogatz (1998). Several empirical studies have identified small-world properties in real world social and physical networks,[2] with possibly the most notorious one being the six degrees of separation: it takes on average six steps to reach any individual in the world [MAYBE HERE WE CITE MILGROM's EXPERIMENT]. This fact is actually a manifestation of a well defined mathematical property, which is a relatively short average path-length. The small-world network model is constructed starting with a regular one-dimensional lattice, and introducing a *rewiring* probability μ based on which any link can be re-wired. Fig. 3 shows examples with $N = 50$ nodes and degree 4 (the total number of links is $4 \times 50/2 = 100$). In the middle panel there is a small-world network where eleven links have been rewired (the rewiring probability was $\mu = 0.1$). The

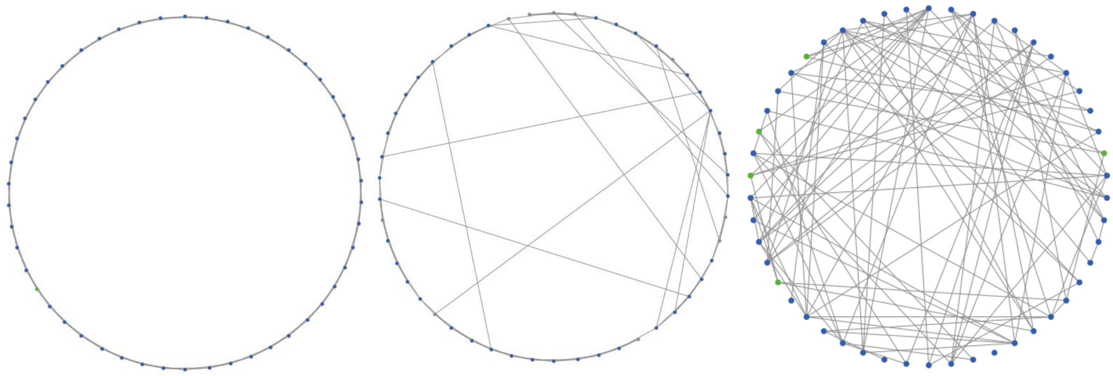


Figure 3: small-worlds construction process: example with 50 nodes and degree 4. Left: regular lattice ($\mu = 0$). Centre: small-world with $\mu = 0.1$. Right: Poisson network ($\mu = 1$).

limit case $\mu = 1$ is a network where all links have been rewired (Fig. 3, right panel), a procedure that leads to a fully random Poisson network of the type introduced by Erdös and Rényi (1960). In this network the connectivity of nodes follows a Poisson distribution. Also Poisson networks are characterized by a relatively short average path-length with respect to other network structures and in particular with respect to the starting regular lattice of Fig. 3, left panel. What makes small-world networks interesting is that they have a short average path-length, comparable with the one of a Poisson network, while preserving another character of the original lattice, which is a high level of clustering.

The clustering coefficient measures the relative number of triplets out of all possible

[2] A review of studies on the statistical properties of real-world networks is Albert and Barabasi (2002).

triplets in a network (Wasserman and Faust, 1994). This fraction is particularly large in the regular lattice, where each node has two neighbours on each side, while it is almost zero in the Poisson network. By rewiring links at random, the average path-length drops suddenly for small values of the rewiring probability, while the clustering coefficient remains practically unaltered until one rewires a large portion of links. The typical small-world of Watts and Strogatz (1998) is obtained with as few as 1% of links rewired. Its average path-length is almost the same of a Poisson network, but the clustering coefficient is very large and comparable to the original regular lattice. It is important to notice that while the rewiring process strongly affects the degree distribution, the average degree remains unchanged, since the numbers of nodes and links are fixed.[3]

We have simulated percolation in a number of different small-world networks, namely for $\mu = 0.001$, $\mu = 0.01$ and $\mu = 0.1$. Fig. 4 reports the results in terms of the final number of adopters as average values over 20 different simulation runs. We observe that a Poisson

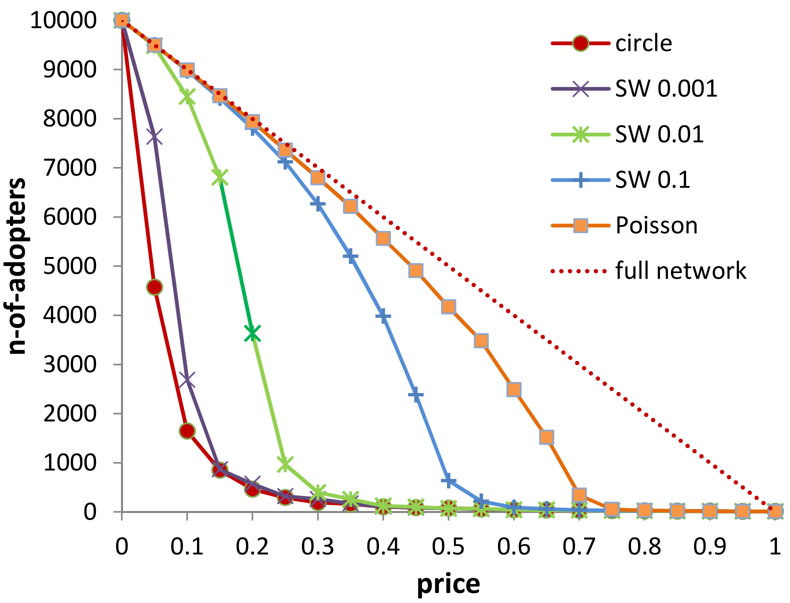


Figure 4: Percolation in different small-world networks: final diffusion size (number of adopters) with 10 initial adopters. Values are averages over 20 runs. Standard deviation is larger at the threshold, and largest for the Poisson network (about 40%). All networks have 10000 nodes (consumers) and an average degree 4.

network represents the best scenario, with largest diffusion size at every price value. The percolation threshold in a small-world is much lower than in a Poisson network, and the non-diffusion regime is much larger. Hence, a much lower price is required for innovation

[3]The degree distribution of a Poisson random network is $p(k) = \frac{1}{k} e^{-z} z^k$, where k is the degree, and the parameter z is the average degree. Each possible link has a probability q such that, given the total number of nodes N, the average degree $z = qN$ is constant (Vega-Redondo, 2007).

to spread in a small-world network compared to a Poisson network. For a typical small-world with $\mu = 0.01$, the price threshold is between 0.2 and 0.3. In the limit case of a regular one-dimensional lattice, the critical price is lowest, between 0.1 and 0.2. The key conclusion here is that small-world networks are highly inefficient compared to Poisson random networks in percolating a new product over a social network of consumers.

The percolation threshold is defined as the critical value of nodes' activation probability where a phase transition occurs in the size of connected components of active nodes, the so-called *percolating cluster*. Newman and Watts (1999) show how to evaluate implicitly the percolation threshold of small-world networks of infinite size. In the price space, the critical value p_c satisfies the following equation:

$$p_c^2 = 4\mu(1 - p_c), \tag{3}$$

where μ is the rewiring probability.[4] In Table 1 we report the values obtained for the small-world networks of Fig. 4. The simulation results of Fig. 4 would present a sharp

	$\mu = 0$	$\mu = 0.001$	$\mu = 0.01$	$\mu = 0.1$	$\mu = 1$
p_c	0	0.06	0.18	0.46	0.67
lost consumers	100%	94%	82%	54%	33%

Table 1: Critical price p_c and inefficiency in small-world networks with different rewiring probability μ. These are theoretical values from Eq. (3), apart from the Poisson network ($\mu = 1$), whose value is from Eq. (1)).

discontinuity at these values in the case of an infinite network. At the threshold p_c the inefficiency from lost demand reaches its maximum level. Such inefficiency is more severe the lower the rewiring probability μ (Table 1). For the typical small-world with $\mu = 0.01$, no less than 82 percent of *willing-to-buy* consumers are lost, because they never get to know about the existence of the new product.

The two extreme limits represented by the regular lattice network and the Poisson random network can be given a societal interpretation. A regular network reflects a

[4]Eq. (3) is Equation (30) of Newman and Watts (1999), and holds in the limit $\mu << 1$. In particular it does not hold for the Poisson network ($\mu = 1$). The critical price for the Poisson Network is given by Eq. (1). The results in Newman and Watts (1999) are obtained with a slightly different model, where links are randomly added (not rewired). This modification of the Watts and Strogatz (1998) model is necessary to avoid that one node remains unconnected with positive probability, giving an infinite average path-length. The two models converge as $\mu \to 0$.

society that is “collectivist”, with all nodes having the exact same degree, and with a high clustering coefficient indicative of social cohesion. The fully random network instead corresponds to an “individualistic” society, with some nodes having higher degree than others, and with a low clustering indicative of a lack of cohesion, but short distances. Real social networks may be closer to one or the other limit of the model (which makes the small-world model so relevant), depending of the specific case considered. For instance, less developed countries are considered to be characterized by a collectivist society, while individualistic societies are typical of industrialized countries (Fogli and Veldkamp, 2014).

4 Scaling patterns of percolation in small-worlds

To some, the low level of network efficiency of small worlds may come as a surprise. Indeed, small worlds are generally considered very good diffusion vehicles, given their short path lengths. However, such an assessment is based on the speed of diffusion, and not on the extent of diffusion. Only in the unlikely case that everyone has a maximum reservation price, the operational network would coincide with the social network itself, and full diffusion would always be realised. In such instances, the speed of diffusion remains the sole relevant performance criterion.

Following the standard downward-sloping demand curve, we instead assume random reservation prices, in particular, uniformly distributed prices. In such instances, we can express network efficiency as the critical price below which full diffusion occurs. In this context, as reported by Figure 4, we observe a relatively low improvement of diffusion size when comparing a regular network to a small-world network. The typical small-world with rewiring probability $\mu = 0.01$ (low average path length and high clustering coefficient) has a critical price which is only marginally larger than the regular network. Only for larger values of the rewiring probability, leading to lower levels of clustering, we find networks becoming more efficient. We thus claim that whenever diffusion is driven by a percolation mechanism, the important factor is a low clustering coefficient, not a short average path-length. We see in Figure 4 that for price values where there is sizable diffusion in a Poisson random network, small-world networks present a much smaller diffusion size. The latter is correlated with the ‘absence’ of clustering (Watts and Strogatz, 1998), and not at all with the average path length.

The key factor in percolation concerns "spreading", that is, low clustering turns into a topological "spreading" of the network, where the number of neighbours increases with distance. In a random network the number z_r of neighbours at distance r is given by (Newman et al., 2001)

$$z_r = \left[\frac{z_2}{z_1}\right]^{r-1} z_1. \tag{4}$$

Connectivity spreads whenever the second order neighbours are in larger number than direct neighbours, $z_2 > z_1$. In general the number of second order neighbours depends on the variance of the connectivity k, $z_2 = \langle k^2 \rangle - \langle k \rangle$ (Vega-Redondo, 2007). Then connectivity spreads in a network whenever $\langle k^2 \rangle > \langle k \rangle$. For Poisson networks $\langle k^2 \rangle = \langle k \rangle^2$, so this condition simplifies to $\langle k \rangle > 1$. Considering that in a Poisson network $\frac{z_2}{z_1} = \langle k \rangle$, and the expected number of neighbours at distance r is $\langle k \rangle^r$. In our model of innovation diffusion we set the average connectivity to $\langle k \rangle = 4$, and we obtain the following value for the expected number of neighbours at distance r in a Poisson network:

$$z_r^{Poisson} = 4^r. \tag{5}$$

This is to be compared with the regular one-dimensional lattice. Here the connectivity is constant, $k = 4$, and also at distance r we have:

$$z_r^{Circle} = 4. \tag{6}$$

small-world networks present non-trivial scaling properties for connectivity, and different regimes within the bounds represented by Eq. (5) and Eq. (6). Newman and Watts (1999) find two regimes.[5] For a given density of rewired links μ, when two nodes are close to each other on the circular reference system, their average distance scales linearly with their relative coordinate and with the network size. When they are far apart and for large networks, their average distance scales logarithmically. These two regimes are separated by a *characteristic distance* $\xi = \frac{1}{2\mu}$, which is defined as the typical distance on the circular reference system between the two ends of a short-cut (rewired link).[6] Below

[5] As already pointed out in Section 3, we use analytical results obtained for the slightly different small-world model where links are randomly added and not-rewired. The reason is purely technical, since rewiring can lead to an infinite average path-length with positive probability, and for low values of rewiring probability μ (the range of interest) the two models coincide.

[6] For the expression of the characteristic distance ξ and for the expression of the surface $A(r)$, we adapt equation (9) of Newman and Watts (1999) to our model setting, where the original lattice has a connectivity range equal to 2, that is two neighbours on either sides.

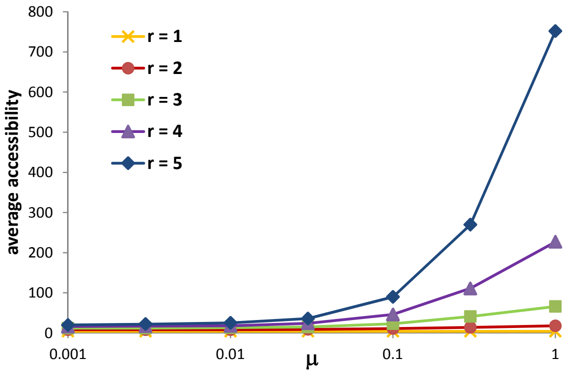
ξ the network presents characteristics close to the regular lattice, and above ξ it is close to the Poisson network. The average number of neighbours at distance r is given by the 'surface' $A(r)$, which is defined as the number of nodes at distance r from any given node.

Adapting the computation of Newman and Watts (1999) to our model setting, we find that in a small-world this number is given by $A(r) = 4e^{8\mu r}$. The exponential scaling of the surface holds when the distance is comparable with the characteristic distance ξ. At short distances $r << \xi$, the surface scales linearly. Resuming, we can express the number of neighbours at distance r in a small-world as follows:

$$z_r^{SW} = \begin{cases} 4(1 + 4r/\xi) & \text{if } r << \xi, \\ 4e^{4r/\xi} & \text{if } r >> \xi. \end{cases} \tag{7}$$

When $\mu << 1$, as in a typical small-world, the regime $r << \xi$ is much more important. For instance, with $\mu = 1\%$ the characteristic distance is $\xi = 50$. This means that neighbours of orders up to $r = 50$ scale only linearly, and very slowly (since the proportionality coefficient is μ). This regime is the one that counts when diffusion works as percolation. Small values of r are crucial, because the surface $A(r)$ at short distances is more affected by inactive nodes (those for which $p_i < p$) than at long distances. Inactive nodes are uniformly distributed, and while at long distances information can easily find a way through the surface, at short distances it may stop easily after a few steps. Only when the rewiring probability is large the surface enlarges fast enough to overcome this hurdle, because the characteristic distance of the small-world is much reduced. For instance, when $\mu = 0.5$ the surface starts to scale up exponentially already after $\xi = 2$ steps.

The volume z_r is a measure of *accessibility* of a network, that is 'how many' nodes can be reached from a given node after r steps. We evaluate the average accessibility for different networks. Fig. 5 shows how z_r scales with the rewiring probability μ at different distances r. The scaling pattern of accessibility correlates perfectly with the absence of clustering (Watts and Strogatz, 1998). The right panel of Fig. 5 shows that in small-world networks accessibility scales linearly, at least for the first steps r, while in a Poisson network it scales exponentially, in agreement with Eq. (7). In a small-world with $\mu = 0.1$ we have $\xi = 5$, and already after five first steps we exit the linear scaling regime. For a small-world with $\mu = 0.01$ the characteristic distance is $\xi = 50$, and we have linear scaling as long as $r < \xi/4 = 12.5$. In the case of a small-world with rewiring probability $\mu = 0.001$, we have $\xi = 500$, and a linear scaling regime endures for more than 100 steps.



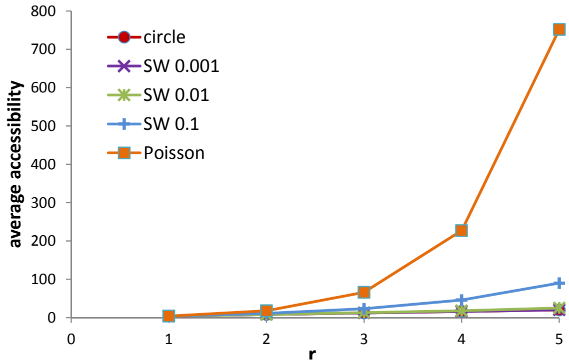


Figure 5: Average accessibility of nodes in small-world networks ($N = 10000$ nodes, average connectivity 4). Left: nodes accessibility as a function of the rewiring probability μ (horizontal axis, logarithmic scale), for different distance values r. Right: nodes accessibility as a function of distance r, for different small-world networks.

In order to understand the effect of the rewiring mechanism on percolation, consider a regular lattice with connectivity 4, and see how the number of neighbours at a given distance r (the surface of radius r) changes for a node with a rewired link. Before rewiring, the node has four r-order neighbour. If one link is rewired, one of these four neighbours is lost, and four new r-order neighbours are found. In Figure 6 we give the example with $r = 2$. Assuming that in r steps we do not find another rewired link, the surface of radius

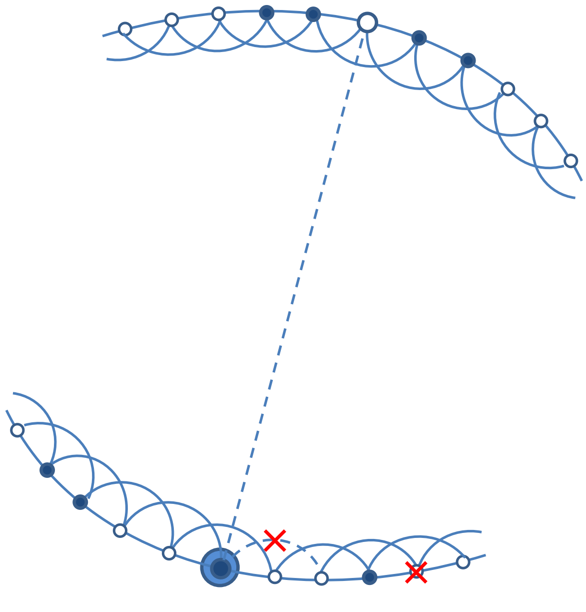


Figure 6: Surface of radius $r = 2$ centered on a node with a rewired link. The surface is made of 4 nodes before rewiring, and 7 nodes after.

r increases from four to seven. In case of a mechanism of random links addition instead of rewiring, the surface increases by a factor 2, from four to eight. However, in both cases only the surfaces of the (few) nodes with rewired links increase, which are $\mu\frac{kN}{2}$. The surface of other nodes is unaffected.

In Table 2 we report the value of the surface of a relatively small radius, $r << \xi$, in the case of a small-world with average connectivity k. We consider nodes interested by the rewiring of a link and 'normal' nodes, and we compare these measures with the slightly different small-world model where links are added, instead of rewired. We then compute the average value of the surface, based on the probability μ that a link is rewired, and also give the value of the volume of the sphere of radius r, which is simply the 'integral' of the surface and which counts the number of nodes at distance equal or smaller than r from a given node. We consider first the case of links rewiring. In a small-world with

	rewiring	adding
r-surface for "normal" nodes	k	k
r-surface for "affected" nodes	$2k-1$	$2k$
average r-surface for any node	$k(1+\mu)-\frac{2}{k}\mu$	$k(1-\mu)$
average r-volume for any node	$r[k(1+\mu)-\frac{2}{k}\mu]$	$rk(1-\mu)$

Table 2: Number of neighbours at distance r (surface) and within distance r for nodes interested (affected) or non interested (normal) by the random rewiring or addition mechanism in a small-world with connectivity k and link rewiring or addition probability μ (assuming $r << \xi = \frac{1}{2\mu}$).

initial connectivity k, the surface of radius r for a node with a rewired link depends on which of its k links has been rewired. If this is a link covering a distance smaller than the *range* $\rho = \frac{k}{2}$ the surface of radius $r < \rho$ is unaffected (one reachable node is lost, one is gained). But if the rewired link was covering a distance equal to the range ρ, the surface changes from k to $2k-1$. Since there are $\frac{2}{k}$ of such 'long range' links for a node with connectivity k, on average the surface of radius r changes with rewiring from k to $2k-\frac{2}{k}$. Considering that only a fraction μ of links are rewired, the surface of radius r of an average node is $r[(1-\mu)k+\mu(2k-\frac{2}{k})] = k(1+\mu)-\frac{2}{k}\mu$. In the case of a small-world model with link addition these measures are way more simple, as reported in Table 2.

In a small-world, being $\mu << 1$, the average surface of radius $r << \xi$ is k, both for the 'rewiring' and the 'adding' links small-world model. The surface is almost constant in r, and connectivity does not 'spread', as it does in the Poisson network. In terms of accessibility, expressed by the volumes of radius r, a small-world is not different from the initial one-dimensional regular lattice, unless the rewiring probability μ is large. To conclude, since percolation is primarily driven by the average accessibility of the network, and given that accessibility of the first few steps is primarily important, due to shut-down nodes, the diffusion size in a small-world is not substantially larger than in the original one-dimensional lattice. Short-cuts do not add substantially to the diffusion size, which remains limited by the low dimensionality of the original lattice.

5 Diffusion time

So far we have been concerned with the size of the diffusion process. Another important aspect of innovation diffusion is the diffusion time. With this we mean the time it takes for the diffusion process to stop. This is the time required to cover all connected components of the operational network that contain a seed. Figure 7 reports the diffusion time for different innovation prices obtained for the simulations already analysed in Figure 4. These results present the typical pattern of second order phase transitions, with a time

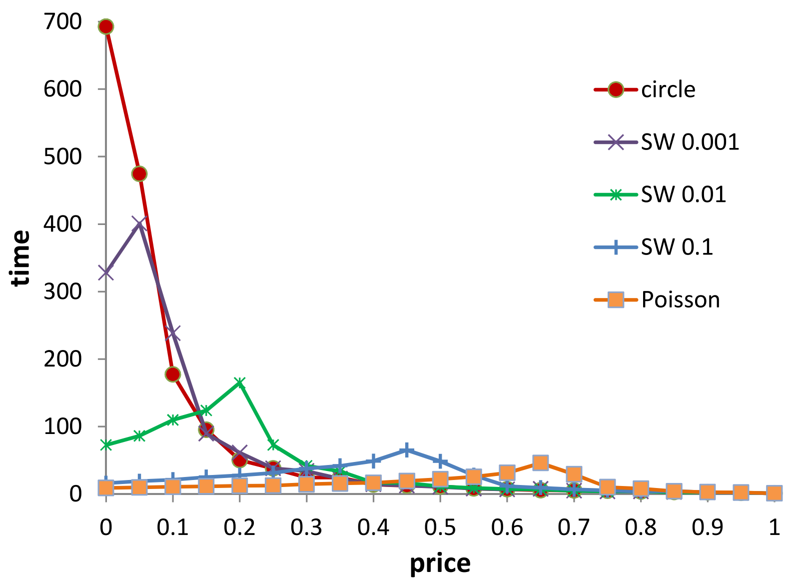


Figure 7: Diffusion time (steps) for percolation in small-world networks with 10000 nodes (consumers) and average degree 4. Values are averages over 20 runs. The standard deviation is larger at the percolation threshold (30%).

peak at the percolation threshold. The latter is an unambiguous way to locate the critical price $p = p_c$ of the transition between the diffusion and no-diffusion regimes. The positions of the peaks, 0.05, 0.2, 0.45 and 0.65, are in agreement with the theoretical values of Table 1. The diffusion time at $p = 0$ and $p = p_c$ from the simulations of Figure 7 are reported in Table 3. The critical price $p = p_c$ is a "worst condition" for diffusion time, where only one path connects several nodes in the network. Above the critical price, diffusion stops very

diffusion time	$\mu = 0$	$\mu = 0.001$	$\mu = 0.01$	$\mu = 0.1$	$\mu = 1$
$p = 0$	693	328	73	16	9
$p = p_c$	693	401	165	65	46

Table 3: Diffusion time at $p = 0$ and at the percolation threshold $p = p_c$ for different networks with 10000 consumers and average degree 4. μ is the rewiring probability. Values are averages over 20 simulation runs.

quickly. On the contrary, $p = 0$ is a "best condition", where all the network is accessible (apart from unconnected components without a seed).

The regular lattice reports the longest diffusion time, as expected, and $p = 0$ is exactly the percolation threshold. The rewiring mechanism reduces distances in the network, and drives down the diffusion time both at $p = 0$ and at the threshold. The important aspect to consider here is how the diffusion time scales with rewiring. There are clearly two regimes: for small rewiring probabilities, below $\mu = 0.01$, the diffusion time scales down fast. Near $\mu = 0.01$, the diffusion time is already relatively short, and further reductions from increasing the rewiring probability are negligible. In Fig. 8 (left panel) we plot the diffusion time values of Table 3 together with the inefficiency measure of Table 1. The two patterns contrast remarkably with each other: the network inefficiency decreases

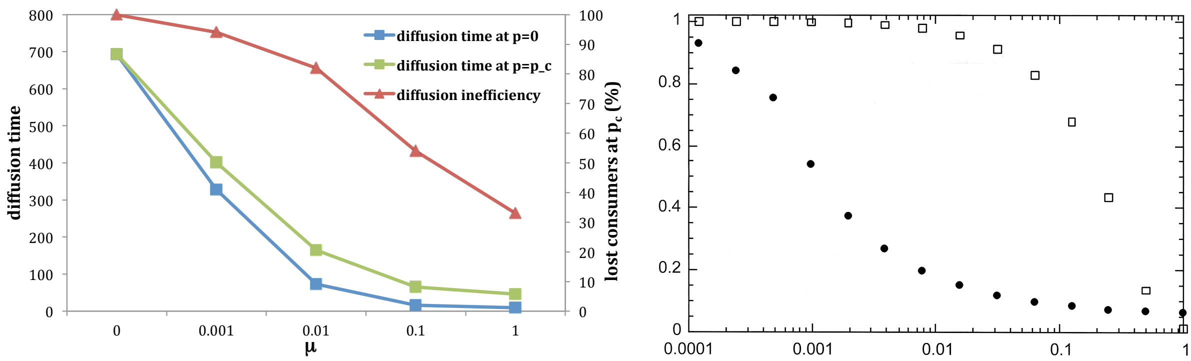


Figure 8: Left: Diffusion time (primary axis) and diffusion inefficiency (secondary axis) for percolation in small-world networks with different rewiring probability μ (horizontal axis). Values are from the simulations of Fig. 7. Inefficiency is computed as $(1 - p_c)$%. Right: Clustering coefficient (squares) and average path-length (dots) in small-worlds from Watts and Strogatz (1998). Values are normalised to the case of the regular lattice ($\mu = 0$).

little below $\mu = 0.01$, and drops fast beyond this value. Such different scaling patterns mean that diffusion size and diffusion time are driven by different network factors. Fig. 8 compares percolation results to structural properties of small-worlds networks such as the clustering coefficient and the average distance between nodes (Watts and Strogatz, 1998). The similarity between the two sets of measures is evident. While diffusion inefficiency (a negative measure of diffusion size) correlates with clustering, diffusion time correlates with average path-length. For low values of μ the average path-length decreases linearly as $\frac{l}{N} = \frac{1}{4} - \frac{1}{2}\mu N + O(\mu^2)$. When $\mu = 0$ we have the average path-length of a circle $l = N/4$, while for large values of μ the approximation above does not hold anymore, and we have $l \simeq \log N$ instead, as in a Poisson network.

The diffusion time at $p = 0$ and $p = p_c$ are linked to different network properties. At $p = 0$ the diffusion time scales with rewiring faster than diffusion time at $p = p_c$ (Table 3). At $p = 0$ the operational network coincides with the full network, and many alternative paths are available to reach any node starting from the seeds of the percolation process. The diffusion time at $p = 0$ depends on the shortest one, and the lower the average path length, the shortest the diffusion time. At the percolation threshold a different scenario realizes. The connected operational component is relatively large, but very often only one path connects distant parts of this network. The diameter of the giant component closely relates to diffusion time in this case, since it is the distance between the two nodes most far away.

6 Alternative demand curves

So far we have used a uniform distribution of reservation prices $p_i \sim U[0,1]$ across consumers (nodes). Given the innovation price $p \in [0,1]$, the uniform distribution gives a linear "prior" or "potential" demand $D(p) = 1-p$. In case of full information, the fraction $1-p$ of consumers adopt the innovation, on average.

In general, for a distribution $f[0,1]$ of reservation price values, the demand with full information is $D(p) = N \times Prob(adoption)$, where

$$\begin{aligned} Prob(adoption) &= Prob[p_i > p] \\ &= 1 - Prob[p_i < p] \\ &= 1 - \int_0^p f(x)dx = 1 - F(p). \end{aligned} \tag{8}$$

$F(p)$ is the cumulative distribution function, so that $D(p) = N[1 - F(p)]$. In this section we study percolation with non-uniform distributions of reservation prices. In particular, we want to understand how the percolation mechanism depends on the potential demand.

Let us consider a Beta distribution of reservation price values, for which the probability density function reads as follows:

$$f(p;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}, \quad p \in [0,1]. \tag{9}$$

The factor $\frac{1}{B(\alpha,\beta)}$ is a constant, defined by $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}$. The parameters α and β control the probability distribution, whose density function can be increasing, decreasing or non-monotonic. Accordingly, the cumulative distribution function $F(p)$ and the resulting demand curve can be convex, concave or S-shaped (Figure 9).

We run batch simulations with different demand curves for three different network structures, namely the regular one-dimensional lattice, a small-world with rewiring probability $\mu = 0.01$ and a Poisson random network. We first consider four cases with different mean value of the individual reservation price. The results reported in Fig. 9 clearly show how the potential demand curve (i.e. the distribution of reservation prices) matters for innovation diffusion in a network of consumers. For a given network structure, different

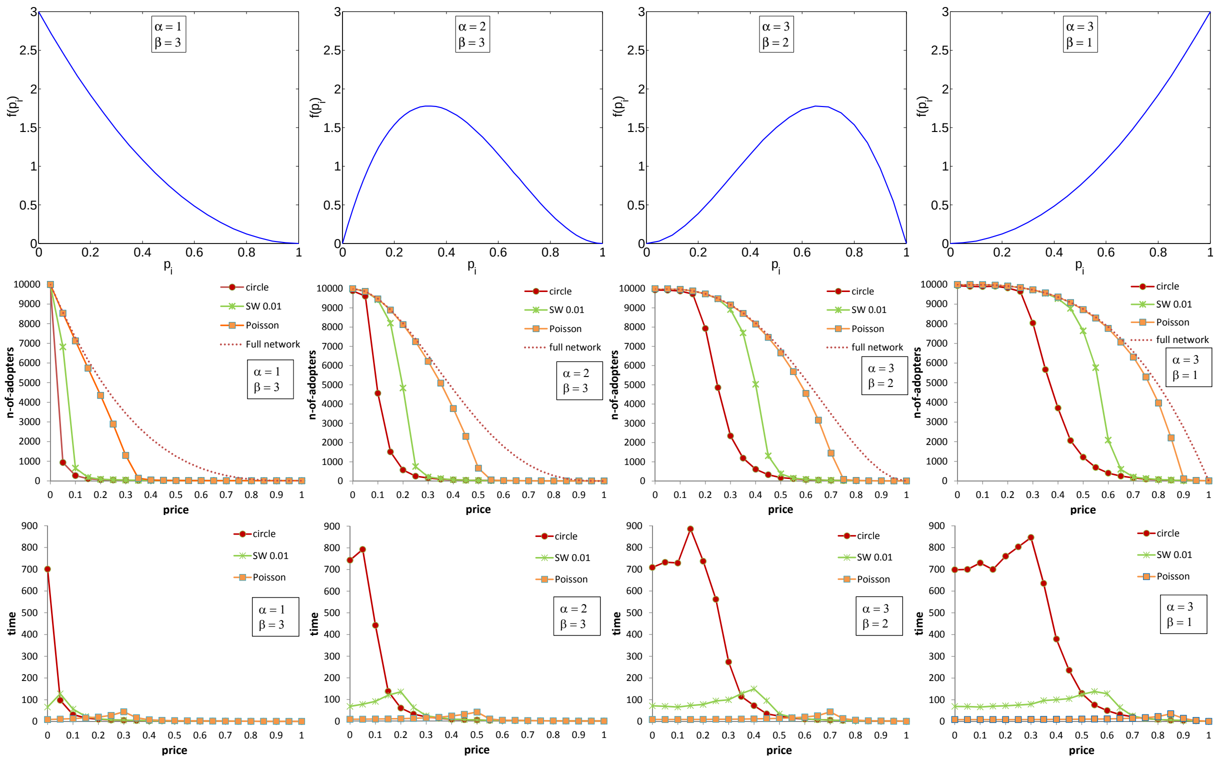


Figure 9: Percolation with alternative demand curves: Beta distribution of reservation prices p_i. Cases with different mean. Top: probability density function. Middle: diffusion size (averages over 20 simulation runs. The standard deviation is larger at the threshold, and increases as the distribution moves to the right, being 50% for $\alpha = 3$, $\beta = 1$). The dashed line is the demand in a fully connected network, $D(p) = N(1 - F(p))$, with F the cumulative distribution of Eq. 8. Bottom diffusion time (averages over 20 simulation runs. The standard deviation is larger at the threshold, and ranges from 25% to 30%). The networks are a one-dimensional regular lattice, a small-world with rewiring probability $\mu = 0.01$, and a Poisson network, with 10000 nodes and (average) connectivity 4.

demand curves lead to different percolation thresholds. Consider for instance a Poisson network. This one has a percolation threshold $p_c \simeq 0.67$ with linear demand (Section 2, Eq. 1 and Figure 2). If we use a decreasing probability density function and the resulting convex demand of the example in the left panel of Figure 9 (case $\alpha = 1$, $\beta = 3$), we obtain a much smaller diffusion regime, with a percolation threshold between $p = 0.3$ and 0.4. An increasing distribution of reservation prices, with a concave demand (right panels of Figure 9) leads to a larger diffusion regime, instead, with a percolation threshold near to $p = 0.9$. Non-monotone probability distributions (middle panels of Figure 9) give a S-shaped potential demand, with a concave and a convex region. When consumers are embedded in a network and innovation diffuses as a percolation process, the larger the mode of the distribution, the larger the percolation threshold and the smaller the inefficiency effect of lost demand due to information transmission (Section 2). The considerations above hold also for the the regular lattice and the small-world: whenever the probability distribution of reservation prices puts more weight on lower values, the percolation threshold decreases, and the diffusion regime shrinks.

The information inefficiency and the loss in demand of a percolation process strongly depend on the reservation price distribution, and for a given distribution they depend on the consumer network structure. For the Poisson network, the more a distribution puts weight on large values of reservation prices, the smaller the inefficiency effect, and the less demand is lost (Figure 9, right panel). The more weight on low reservation price values, and the less efficient the diffusion process, instead, with a larger loss of demand (Figure 9, left panel). For the regular lattice, the opposite is true: when there is more weight on larger reservation prices, the realized demand increases, with a larger threshold, but not as much as for the Poisson network, and the difference with the potential demand is amplified. The reason is as follows: a larger mass of consumers with low reservation price produces many bottlenecks in the network, which reduce information efficiency and lead to a low diffusion regime (Section 2). A larger mass of consumers with high reservation prices enlarges the connected component of the operational network. This effect is much stronger the higher the dimensionality of the network, when redundant links are rewired and used to open alternative routes that spread from the seeds of the diffusion process.

The percolation threshold gives an absolute measure of diffusion: the "fatter" the potential demand is, the larger the diffusion regime. The loss in demand is a relative measure instead, which tells about the efficiency of the network structure. When the potential demand gets "fatter" the Poisson network experiences a smaller loss of demand. However, inefficient network structures such as the circle or the small-world are less able to exploit this more favourable distribution of reservation price. If we think of the reservation price as a manifestation of consumers' income, this means that in a more rich society the diffusion inefficiency due to the network structure are amplified.

The diffusion time is not affected by changes in the distribution of reservation prices, apart from the shift of the threshold peak (Fig. 9, bottom panels). Both at $p = 0$ and $p = p_c$ the diffusion time does not change much for different distributions. The results for the one-dimensional lattice are interesting. The percolation threshold shifts to the right with distributions that have more probability mass at larger values of price, but a peak for the diffusion time is missing: the diffusion time remains long also well below the threshold. The reason is that in a one-dimensional lattice there are only at most four alternative paths to cover the network, and diffusion becomes a linear process. The larger is the distance to cover, the longer the time required, and diffusion time becomes proportional to diffusion size.

We have studied the effect of the dispersion of reservation prices by considering symmetric distributions with mean preserving spread (Fig. 10). When the distribution is less dispersed (and its peak more pronounced), meaning that consumers are more alike, the critical price of different network structures converges to the mean value of the distribution ($p = 0.5$). Put differently, diverse network structures look more similar for their diffusion outcomes. When reservation prices are narrowly distributed, the mean becomes a critical value for the innovation price, above which almost all network nodes are shut down, and above which almost all of them are accessible. In the limit case of a homogenous reservation price (δ-Dirac distribution located in $p = 0.5$), the demand curve is a step function, the mean reservation price 'attracts' the critical prices, and different networks present almost the same diffusion size. This is quite the case with the example in the right panels of Fig. 10: irrespective of the network structure, there is a diffusion regime on $[0, 0.5]$, and a no-diffusion regime on $[0.5, 1]$. In this limit the percolation process is able to get the full potential demand for all network structures, with no lost demand from information inefficiency.

The latter results have obvious implications for consumers' welfare. First, the convergence of different network structure towards a single reservation price results in a net gain

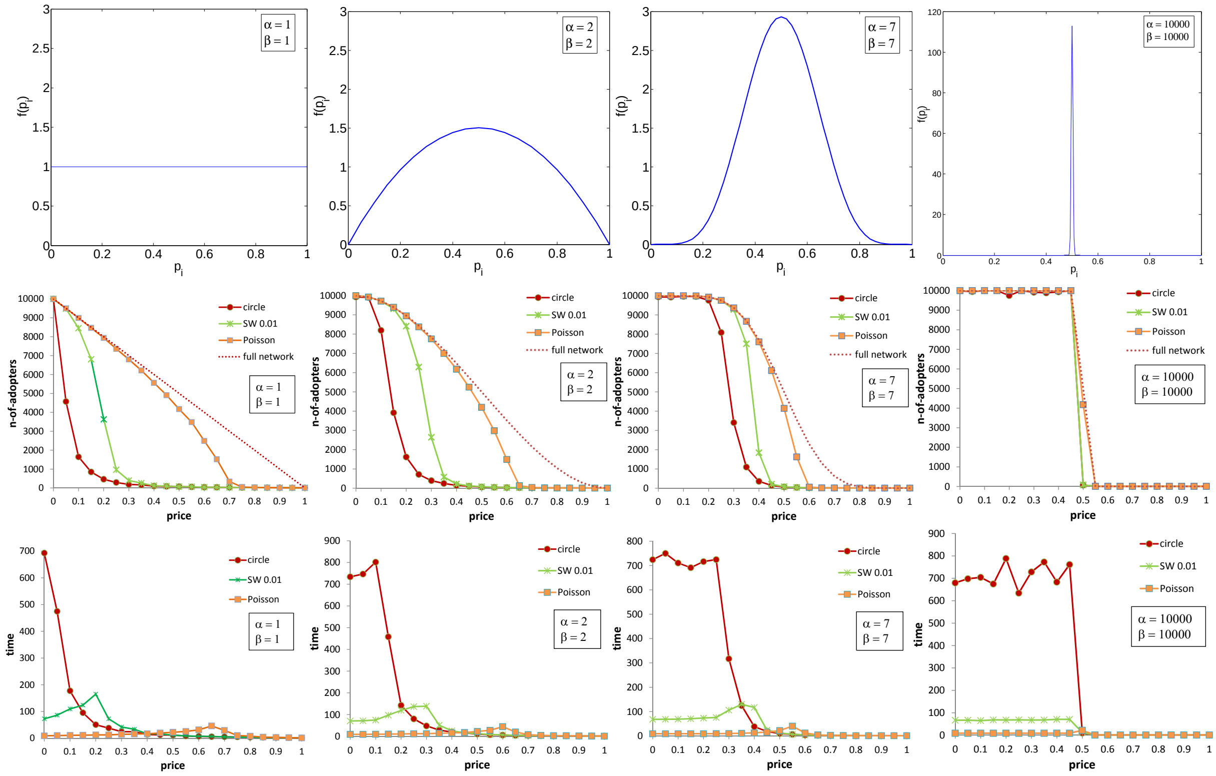


Figure 10: Percolation with alternative demand curves. Beta distribution of reservation prices. Cases with equal mean ($< p_i >= 0.5$). Top: probability density function. Middle: diffusion size (averages over 20 runs. The standard deviation is larger at the threshold, and increases as the distribution moves to the right, being 50% for $\alpha = 3$, $\beta = 1$). The dashed line is the demand in a fully connected network, $D(p) = N(1 - F(p))$, with F the cumulative distribution of Eq. 8. Bottom diffusion time (averages over 20 simulation runs. The standard deviation is larger at the threshold, and ranges from 25% to 30%). The networks are a one-dimensional regular lattice, a small-world with rewiring probability $\mu = 0.01$, and a Poisson network, with 10000 nodes and (average) connectivity 4.

of consumers' surplus. Less efficient networks like the regular lattice and the small-world network gain both in terms of a higher critical price and a smaller loss of consumers with respect to the potential demand. But also for the Poisson network, the effect of a smaller critical price is outweighed by a larger area in the diffusion regime, with reduced size of consumers loss.

7 Scale-free networks

Real social networks often present a "hub" structure, which is by no means captured by small-worlds and Poisson random networks. Few nodes, the hubs, have many links, while the majority of nodes only have few links. Examples are the World-Wide-Web, the internet, science collaboration networks, and many others (Albert and Barabasi, 2002). This network structure is characterized by a power law distribution of the degree. A power law distribution is also called "scale-free": for any value of the degree, the probability of occurrence of nodes with such degree "scales" down with the degree at the same rate. This means that if on average there are 10 nodes with 1000 links, we may expect to find 100 nodes with 100 links, 1000 nodes with 10 links, and so on.

The scale-free network model introduced by Barabasi and Albert (1999) is essentially an algorithm to generate a graph with a power law degree distribution. The basic idea is a self-reinforcement mechanism of link creation, which builds on two factors, growth and preferential attachment. At each time step a new node is added to the network, and linked to existing nodes with a probability which is proportional to their degree (an instance of rich-get-richer positive feedback). The model has some variants, as for instance the possibility to add more than one link for a new node. In this case the network can have a triadic structure, while a tree-like structure is generated if only one link is introduced for each new node (Figure 11, right panel). In general, if we grow a network with this algorithm, starting with N_0 nodes and adding m new links for each new node until we have $N = mt + N_0$ nodes, the final number of links is as follows:

$$N_{links} = mt - \frac{(m - N_0)(m - N_0 + 1)}{2}, \tag{10}$$

which equals approximately mt links as soon as $t >> m$. The connectivity k of such network is distributed with a probability density function $p(k) \sim k^{-\gamma}$ where $\gamma = 2.9 \pm 0.1$. The striking property of such scale-free networks is that $p(k)$ does not depend on time t and on size $N = mt + N_0$: for any starting number of nodes N_0 and any rate of links addition m, at each time step t the connectivity distribution has the same shape γ. The mean of such distribution is time invariant, but it does depend on the initial number of nodes N_0 (although it is independent on m):

$$\langle k \rangle = \frac{\gamma - 1}{\gamma - 2} k_{min}, \tag{11}$$

where k_{min} is the lower bound of the distribution support. If we add m new links at each time step, such lower bound is exactly $k_{min} = m$. In a scale-free network generated with $m = 1$ then we have $\langle k \rangle \simeq 2$, while for $m = 2$ we have $\langle k \rangle \simeq 4$. Figure 11 shows an example of scale-free network with $m = 2$ and average degree 4. This realization presents some hubs on the lower-right part of the network.

We run a number of simulations to see how percolation works on scale-free networks. Fig. 12 compares the diffusion size and the diffusion time in a scale-free network with

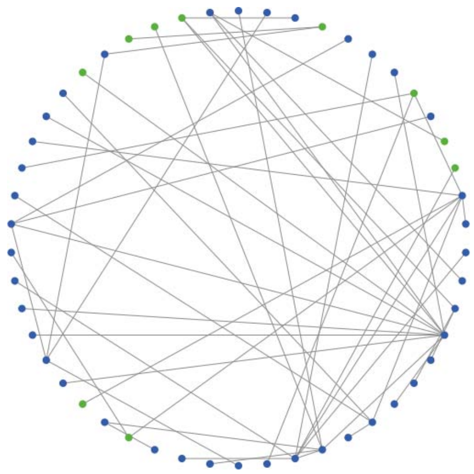


Figure 11: Scale-free network generated adding $m = 2$ links for each new node, up to $N = 50$ nodes.

a small-world network and a Poisson random network, all with average degree $\langle k \rangle = 4$. The results in the left panel show that scale-free networks are relatively efficient in terms of diffusion size, and roughly match the the demand pattern of consumers arranged in a Poisson network. This is in accordance with the low degree of clustering of both structures. The scale-free network gives a smoother transition between diffusion and no-diffusion regimes. Its critical price is larger than the Poisson network critical price, meaning that scale-free networks favour diffusion when the price is relatively high. Below the threshold the Poisson network is slightly more efficient. The reason is that hubs are useful when the innovation price is relatively high, because whenever a hub adopts the innovation, it passes the information to many neighbours, and very likely find some with reservation price high enough to adopt the innovation. However, when the price is relatively low, this advantage is less important, and the high-dimensionality of a Poisson network is more important.

The diffusion time dimension is also interesting, and show that percolation processes in scale-free networks are relatively fast. On average, they report diffusion times lower than the Poisson network and the small-world network, both at the critical transition threshold and below it (Figure 12, right panel). In particular, a low diffusion time at the threshold reflects the smoothness of the critical transition of the diffusion size.[7]

[7]In a scale-free network with $\langle k \rangle = 2$, generated adding one link for a new node, the percolation threshold is $p_c = 0$, and the relationship between size and time of diffusion is monotone: a larger diffusion size requires a longer diffusion time. This is a consequence of the tree-like structure of this scale-free network, where there is only one path from a seed to any node.

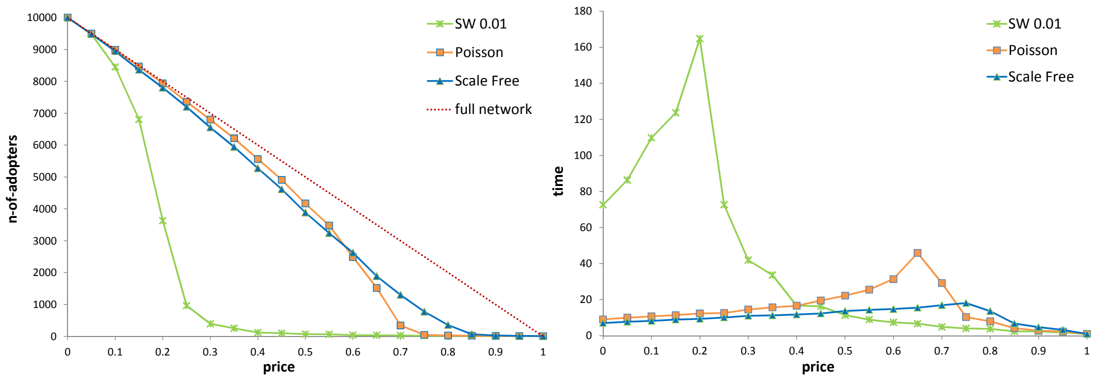


Figure 12: Percolation in a Scale Free network (built with two new links for each new node), a Small World ($\mu = 0.01$) and a Poisson network. Left: diffusion size. Right: diffusion time. All networks have 10000 nodes andaverage degree 4, and percolation experiments start with 10 seeds (early adopters). Values are averages over 20 simulation runs.

8 Conclusions

A percolation model combines two important factors of innovation diffusion, which are adoption decisions and information spreading. Percolation shows a phase transition from a diffusion to a no-diffusion regime (phase), for increasing prices. The phase transition indicates the critical price below which diffusion is almost complete. We show that the critical price can be used as a measure of network efficiency, which depends on network topology and the distribution of reservation prices.

Percolation processes in innovation diffusion have two economic implications. First, it highlights an instance of network inefficiency: a sizeable portion of the demand is not satisfied in the no-diffusion phase regime. Second, percolation processes unfold differently in different network topologies. In particular, our results challenge the common wisdom according to which small-worlds are favourable for innovation diffusion. We show that whenever diffusion works as a percolation process, small-worlds are rather inefficient, as diffusion size is driven by low clustering, and not by low average path-length of a network. We further show that, apart from low clustering, a less dispersed distribution of reservation prices across consumers favour diffusion, suggesting that not only richer, but also more equal societies, support new product diffusion.

Our key result on the effect of clustering on diffusion size are in line with empirical evidence on technology diffusion (Fogli and Veldkamp, 2014), but against experimental evidence on behaviour diffusion (Centola, 2010). Our percolation model offers a clear benchmark for the adoption mechanism: when innovation adoption is driven by individual preferences only, and links between consumers only carry information, clustering has a negative effect. For behavior spreading, however, it is more likely that social pressures also play a role, in that individuals become more likely to adopt a behavior if many neighbours already display the behavior. An extension of the basic percolation model with a social-pressure function would indeed be a promising line for future research.

There are limitations to the present study, which represent possible directions for further research. First, our analysis of the welfare implications of social networks has been limited to the demand side, focusing on the consumer surplus that is lost when consumer remain unaware of a product which they would be willing to buy. Since, for monopolistic market structures, lower demand translates into lower prices, some of the welfare losses will be compensated by lower equilibrium prices. Campbell (2013) provides such an equilibrium model illustrating the latter effect, but without considering welfare effects due to the loss of consumer surplus (and restricted to random networks only). In future research, percolation theory can be integrated in the welfare analysis of markets, where equilibrium prices are derived, rather than treated as a parameter value.

Second, a rather strong assumption in our model is that links of consumers networks are uncorrelated with reservation prices. Since the latter likely reflect consumers' income, a more realistic description of the diffusion process would require to relax this assumption, and consider some degree of assortativity in network structures, where neighbour consumers are more likely to have similar reservation prices. Assortative networks and income distribution are two key-factors in the study of the implications of percolation for innovation policy.

Third, in our framework social networks acted solely as the medium for information diffusion among consumers, thus ignoring other possible effects of social interactions. For example, consumers may become more likely adopters, the more neighbours already adopted due to local network externalities causing a consumer's willingness to pay to increase with the number of neighbours already adopting. Furthermore, a product's price may fall over time as a function of the total number of adopters, known as learning-by-doing. Such additional mechanisms will affect the extent to which a product diffuse in different networks, as shown in previous models (Delre et al., 2007, 2010).

Indeed, the percolation model is first and foremost a theoretical model of diffusion dynamics driven by individual reservation prices, rather than an accurate description of various diffusion processes that exist in reality. As a null-model based on price and network topology only, it is a useful benchmark in empirical studies of diffusion dynamics. To the extent that empirical data systematically deviate from the predictions of percolation concerning new product diffusion, additional mechanisms, such as the ones described above, should be included to better understand the exact nature of diffusion in specific technological and social contexts.

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