Karlsruhe Institute of Technology

Endogenous Growth Theory

Lecture Notes for the winter term 2010/2011

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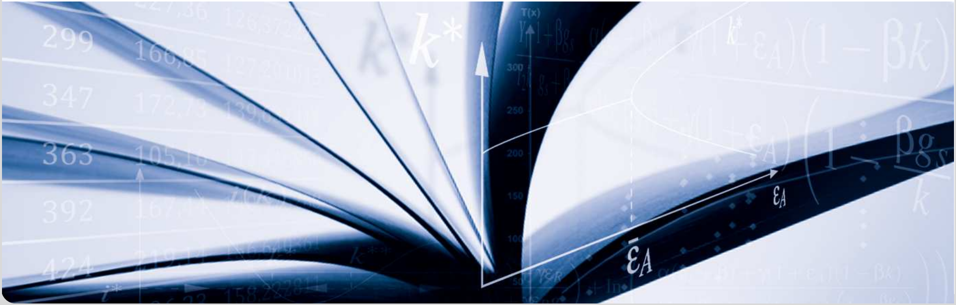
CHAIR IN ECONOMIC POLICY



Solow Growth Model

- Develop a simple framework for the *proximate* causes and the mechanics of economic growth and cross-country income differences.
- Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the *Solow model*
- Before Solow growth model, the most common approach to economic growth built on the Harrod-Domar model.
- Harrod-Domar model emphasized potential dysfunctional aspects of growth: e.g, how growth could go hand-in-hand with increasing unemployment.
- Solow model demonstrated why the Harrod-Domar model was not an attractive place to start.
- At the center of the Solow growth model is the *neoclassical* aggregate production function.

The Economic Environment of the Basic Solow Model

- Study of economic growth and development therefore necessitates dynamic models.
- Despite its simplicity, the Solow growth model is a dynamic general equilibrium model (though many key features of dynamic general equilibrium models, such as preferences and dynamic optimization are missing in this model).

Households and Production I

- Closed economy, with a unique final good.
- Discrete time running to an infinite horizon, time is indexed by $t = 0, 1, 2, ...$
- Economy is inhabited by a large number of households, and for now households will not be optimizing.
- This is the main difference between the Solow model and the *neoclassical growth model*.
- To fix ideas, assume all households are identical, so the economy admits *a representative household*.

Households and Production II

- Assume households save a constant exogenous fraction s of their disposable income
- Same assumption used in basic Keynesian models and in the Harrod-Domar model; at odds with reality.
- Assume all firms have access to the same production function: economy admits a **representative firm**, with a representative (or aggregate) production function.
- Aggregate production function for the unique final good is

$$Y(t) = F[K(t), L(t), A(t)] \tag{1}$$

- Assume capital is the same as the final good of the economy, but used in the production process of more goods.
- $A(t)$ is a *shifter* of the production function (1). Broad notion of technology.
- Major assumption: technology is **free**; it is publicly available as a non-excludable, non-rival good.

Key Assumption

Assumption 1 **(Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale)** The production function F: $\mathbb{R}^3_+ \rightarrow \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K,L,A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K,L,A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0,$$
$$F_{KK}(K,L,A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K,L,A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

Moreover, F exhibits constant returns to scale in K and L.

- Assume F exhibits *constant returns to scale* in K and L. I.e., it is *linearly homogeneous* (homogeneous of degree 1) in these two variables.

Review

Definition Let K be an integer. The function $g : \mathbb{R}^{K+2} \to \mathbb{R}$ is homogeneous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ iff

$$g(\lambda x, \lambda y, z) = \lambda^m g(x, y, z) \text{ for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^K.$$

Theorem **(Euler's Theorem)** Suppose that $g : \mathbb{R}^{K+2} \to \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y and is homogeneous of degree m in x and y. Then

$$\begin{aligned} mg(x,y,z) &= g_x(x,y,z)\,x + g_y(x,y,z)\,y \\ \text{for all } x &\in \mathbb{R}, y \in \mathbb{R} \text{ and } z \in \mathbb{R}^K. \end{aligned}$$

Moreover, $g_x(x,y,z)$ and $g_y(x,y,z)$ are themselves homogeneous of degree $m-1$ in x and y.

Market Structure, Endowments and Market Clearing I

- We will assume that markets are competitive, so ours will be a prototypical *competitive general equilibrium model*.
- Households own all of the labor, which they supply inelastically.
- Endowment of labor in the economy, $\bar{L}(t)$, and all of this will be supplied regardless of the price.
- The *labor market clearing* condition can then be expressed as:

$$L(t) = \bar{L}(t) \tag{2}$$

  for all t, where $L(t)$ denotes the demand for labor (and also the level of employment).
- More generally, should be written in complementary slackness form.
- In particular, let the *wage rate* at time t be $w(t)$, then the labor market clearing condition takes the form

$$L(t) \leq \bar{L}(t), w(t) \geq 0 \text{ and } \left(L(t) - \bar{L}(t)\right) w(t) = 0$$

Market Structure, Endowments and Market Clearing II

- But Assumption 1 and competitive labor markets make sure that wages have to be strictly positive.
- Households also own the capital stock of the economy and rent it to firms.
- Denote the *rental price of capital* at time t by $R(t)$.
- Capital market clearing condition:

$$K^s(t) = K^d(t)$$

- Take households' initial holdings of capital, $K(0)$, as given
- $P(t)$ is the price of the final good at time t, normalize the price of the final good to 1 *in all periods*.
- Build on an insight by Kenneth Arrow (Arrow, 1964) that it is sufficient to price *securities* (assets) that transfer one unit of consumption from one date (or state of the world) to another.

Market Structure, Endowments and Market Clearing III

- Implies that we need to keep track of an *interest rate* across periods, $r(t)$, and this will enable us to normalize the price of the final good to 1 in every period.
- *General equilibrium economies*, where different commodities correspond to the same good at different dates.
- The same good at different dates (or in different states or localities) is a different commodity.
- Therefore, there will be *an infinite number of commodities*.
- Assume capital depreciates, with "exponential form," at the rate δ: out of 1 unit of capital this period, only $1-\delta$ is left for next period.
- Loss of part of the capital stock affects the interest rate (rate of return to savings) faced by the household.
- *Interest rate* faced by the household will be $r(t) = R(t) - \delta$.

Firm Optimization I

- Only need to consider the problem of a *representative firm*:

$$\max_{L(t)\geq 0, K(t)\geq 0} F[K(t), L(t), A(t)] - w(t)L(t) - R(t)K(t). \qquad (3)$$

- Since there are no irreversible investments or costs of adjustments, the production side can be represented as a static maximization problem.
- Equivalently, *cost minimization problem.*
- Features worth noting:
  1. Problem is set up in terms of aggregate variables.
  2. Nothing multiplying the F term, price of the final good has been normalized to 1.
  3. Already imposes competitive factor markets: firm is taking as given $w(t)$ and $R(t)$.
  4. Concave problem, since F is concave.

Firm Optimization II

- Since F is differentiable, first-order necessary conditions imply:

$$w(t) = F_L[K(t), L(t), A(t)], \tag{4}$$

and

$$R(t) = F_K[K(t), L(t), A(t)]. \tag{5}$$

- Note also that in (4) and (5), we used $K(t)$ and $L(t)$, the amount of capital and labor used by firms.
- In fact, solving for $K(t)$ and $L(t)$, we can derive the capital and labor demands of firms in this economy at rental prices $R(t)$ and $w(t)$.
- Thus we could have used $K^d(t)$ instead of $K(t)$, but this additional notation is not necessary.

Firm Optimization III

Proposition Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y(t) = w(t) L(t) + R(t) K(t).$$

- **Proof:** Follows immediately from Euler Theorem for the case of $m = 1$, i.e., constant returns to scale.
- Thus firms make no profits, so ownership of firms does not need to be specified.

Second Key Assumption

Assumption 2 **(Inada conditions)** F satisfies the Inada conditions

$$\begin{aligned}\lim_{K\to 0} F_K(\cdot) &= \infty \text{ and } \lim_{K\to\infty} F_K(\cdot) = 0 \text{ for all } L > 0 \text{ all } A \\ \lim_{L\to 0} F_L(\cdot) &= \infty \text{ and } \lim_{L\to\infty} F_L(\cdot) = 0 \text{ for all } K > 0 \text{ all } A.\end{aligned}$$

- Important in ensuring the existence of *interior equilibria.*
- It can be relaxed quite a bit, though useful to get us started.

Production Functions

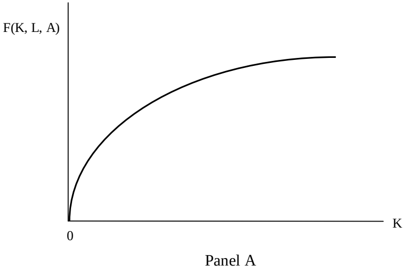


Figure 1.1: Production functions and the marginal product of capital. The example in Panel A satisfies the Inada conditions in Assumption 2, while the example in Panel B does not.

Fundamental Law of Motion of the Solow Model I

- Recall that K depreciates exponentially at the rate δ, so
$$K(t+1) = (1-\delta)K(t) + I(t), \tag{6}$$
where $I(t)$ is investment at time t.
- From national income accounting for a closed economy,
$$Y(t) = C(t) + I(t), \tag{7}$$
- Using (1), (6) and (7), any *feasible* dynamic allocation in this economy must satisfy
$$K(t+1) \leq F[K(t), L(t), A(t)] + (1-\delta)K(t) - C(t)$$
for $t = 0, 1, ...$
- *Behavioral rule* of the constant saving rate simplifies the structure of equilibrium considerably.

Fundamental Law of Motion of the Solow Model II

- Note not derived from the maximization of utility function: welfare comparisons have to be taken with a grain of salt.
- Since the economy is closed (and there is no government spending),

$$S(t) = I(t) = Y(t) - C(t).$$

- Individuals are assumed to save a constant fraction s of their income,

$$S(t) = sY(t), \tag{8}$$

$$C(t) = (1-s)Y(t) \tag{9}$$

- Implies that the supply of capital resulting from households' behavior can be expressed as

$$K^s(t) = (1-\delta)K(t) + S(t) = (1-\delta)K(t) + sY(t).$$

Fundamental Law of Motion of the Solow Model III

- Setting supply and demand equal to each other, this implies $K^s(t) = K(t)$.
- From (2), we have $L(t) = \bar{L}(t)$.
- Combining these market clearing conditions with (1) and (6), we obtain *the fundamental law of motion* of the Solow growth model:

$$K(t+1) = sF[K(t), L(t), A(t)] + (1-\delta) K(t). \tag{10}$$

- Nonlinear *difference equation*.
- Equilibrium of the Solow growth model is described by this equation together with laws of motion for $L(t)$ (or $\bar{L}(t)$) and $A(t)$.

Definition of Equilibrium I

- Solow model is a mixture of an old-style Keynesian model and a modern dynamic macroeconomic model.
- Households do not optimize, but firms still maximize and factor markets clear.

Definition In the basic Solow model for a given sequence of $\{L(t), A(t)\}_{t=0}^{\infty}$ and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that $K(t)$ satisfies (10), $Y(t)$ is given by (1), $C(t)$ is given by (24), and $w(t)$ and $R(t)$ are given by (4) and (5).

- Note an equilibrium is defined as an entire path of allocations and prices: *not* a static object.

Equilibrium Without Population Growth and Technological Progress I

- Make some further assumptions, which will be relaxed later:
  1. There is no population growth; total population is constant at some level $L > 0$. Since individuals supply labor inelastically, $L(t) = L$.
  2. No technological progress, so that $A(t) = A$.
- Define the capital-labor ratio of the economy as

$$k(t) \equiv \frac{K(t)}{L}, \tag{11}$$

- Using the constant returns to scale assumption, we can express output (income) per capita, $y(t) \equiv Y(t)/L$, as

$$\begin{aligned} y(t) &= F\left[\frac{K(t)}{L}, 1, A\right] \\ &\equiv f(k(t)). \end{aligned} \tag{12}$$

Equilibrium Without Population Growth and Technological Progress II

- Note that $f\left(k\right)$ here depends on A, so I could have written $f\left(k, A\right)$; but A is constant and can be normalized to $A = 1$.
- From Euler Theorem,

$$\begin{aligned} R\left(t\right) &= f'\left(k\left(t\right)\right) > 0 \text{ and} \\ w\left(t\right) &= f\left(k\left(t\right)\right) - k\left(t\right) f'\left(k\left(t\right)\right) > 0. \end{aligned} \tag{13}$$

- Both are positive from Assumption 1.

Example: The Cobb-Douglas Production Function I

- Very special production function and many interesting phenomena are ruled out, but widely used:

$$\begin{aligned} Y(t) &= F[K(t), L(t), A(t)] \\ &= AK(t)^{\alpha} L(t)^{1-\alpha}, \ 0 < \alpha < 1. \end{aligned} \tag{14}$$

- Satisfies Assumptions 1 and 2.
- Dividing both sides by $L(t)$,

$$y(t) = Ak(t)^{\alpha},$$

- From equation (13),

$$\begin{aligned} R(t) &= \frac{\partial Ak(t)^{\alpha}}{\partial k(t)}, \\ &= \alpha Ak(t)^{-(1-\alpha)}. \end{aligned}$$

Example: The Cobb-Douglas Production Function II

- Alternatively, in terms of the original production function (14),

$$\begin{aligned} R(t) &= \alpha A K(t)^{\alpha-1} L(t)^{1-\alpha} \\ &= \alpha A k(t)^{-(1-\alpha)}, \end{aligned}$$

- Similarly, from (13),

$$\begin{aligned} w(t) &= A k(t)^{\alpha} - \alpha A k(t)^{-(1-\alpha)} \times k(t) \\ &= (1-\alpha) A K(t)^{\alpha} L(t)^{-\alpha}, \end{aligned}$$

Equilibrium Without Population Growth and Technological Progress I

- The per capita representation of the aggregate production function enables us to divide both sides of (10) by L to obtain:

$$k(t+1) = sf(k(t)) + (1-\delta)k(t). \tag{15}$$

- Since it is derived from (10), it also can be referred to as the *equilibrium difference equation* of the Solow model
- The other equilibrium quantities can be obtained from the capital-labor ratio $k(t)$.

Definition A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $k(t) = k^*$ for all t.

- The economy will tend to this steady-state equilibrium over time (but never reach it in finite time).

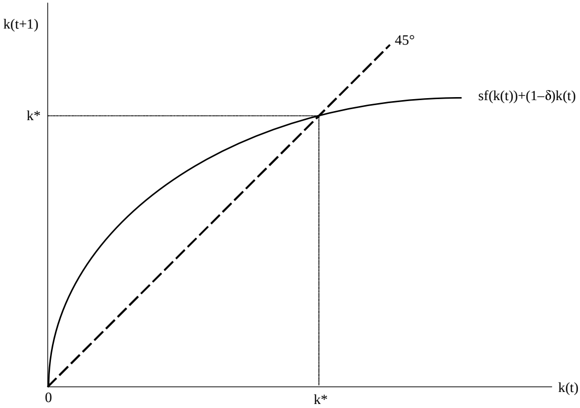


Figure 2.1: Determination of the steady-state capital-labor ratio in the Solow model without population growth and technological change.

Equilibrium Without Population Growth and Technological Progress II

- The thick curve represents (15) and the dashed line corresponds to the 45° line.
- Their (positive) intersection gives the steady-state value of the capital-labor ratio k^*,

$$\frac{f\left(k^*\right)}{k^*} = \frac{\delta}{s}. \tag{16}$$

- There is another intersection at $k = 0$, because the figure assumes that $f\left(0\right) = 0$.
- Will ignore this intersection throughout:
  1. If capital is not essential, $f\left(0\right)$ will be positive and $k = 0$ will cease to be a steady-state equilibrium
  2. This intersection, even when it exists, is an *unstable point*
  3. It has no economic interest for us.

Equilibrium Without Population Growth and Technological Progress III

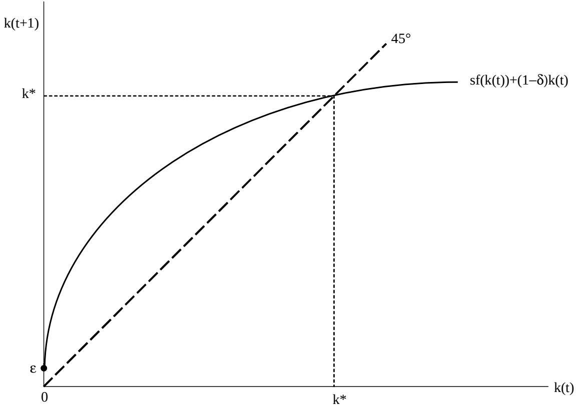


Figure 2.2: Unique steady state in the basic Solow model when $f(0) = \varepsilon > 0$.

Equilibrium Without Population Growth and Technological Progress IV

- Alternative visual representation of the steady state: intersection between δk and the function $sf(k)$. Useful because:
  1. Depicts the levels of consumption and investment in a single figure.
  2. Emphasizes the steady-state equilibrium; sets investment, $sf(k)$, equal to the amount of capital that needs to be “replenished”, δk.

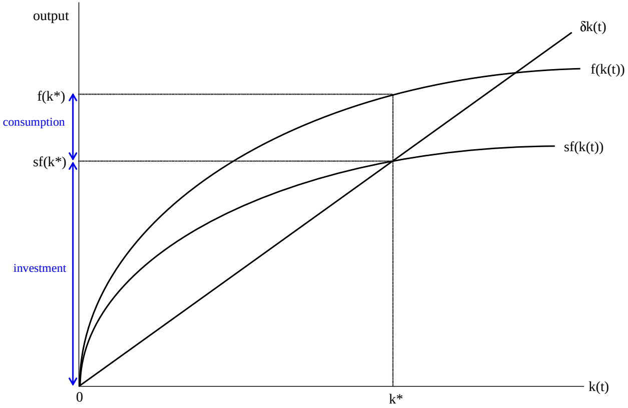


Figure 2.3: Investment and consumption in the steady-state equilibrium.

Equilibrium Without Population Growth and Technological Progress V

Proposition Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady-state equilibrium where the capital-labor ratio $k^* \in (0, \infty)$ is given by (16), per capita output is given by

$$y^* = f(k^*) \tag{17}$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*) . \tag{18}$$

Proof of Theorem

- The preceding argument establishes that any k^* that satisfies (16) is a steady state.
- To establish existence, note that from Assumption 2 (and from L'Hôpital's rule), $\lim_{k\to 0} f(k)/k = \infty$ and $\lim_{k\to\infty} f(k)/k = 0$.
- Moreover, $f(k)/k$ is continuous from Assumption 1, so by the intermediate value theorem there exists k^* such that (16) is satisfied.
- To see uniqueness, differentiate $f(k)/k$ with respect to k, which gives

$$\frac{\partial [f(k)/k]}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0, \tag{19}$$

  where the last equality uses (13).
- Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value k^* that satisfies (16).
- Equations (17) and (18) then follow by definition.

Examples

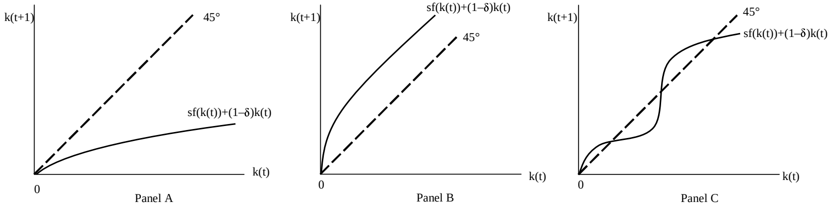


Figure 2.4: Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

Equilibrium Without Population Growth and Technological Progress VI

- Figure 2.4 shows through a series of examples why Assumptions 1 and 2 cannot be dispensed with for the existence and uniqueness results.
- Generalize the production function in one simple way, and assume that

$$f(k) = a\tilde{f}(k),$$

- $a > 0$, so that a is a ("Hicks-neutral") shift parameter, with greater values corresponding to greater productivity of factors..
- Since $f(k)$ satisfies the regularity conditions imposed above, so does $\tilde{f}(k)$.

Equilibrium Without Population Growth and Technological Progress VII

Proposition Suppose Assumptions 1 and 2 hold and $f(k) = a\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(a, s, \delta)$ and the steady-state level of output by $y^*(a, s, \delta)$ when the underlying parameters are a, s and δ. Then we have

$$\begin{aligned}
\frac{\partial k^*(\cdot)}{\partial a} &> 0, \frac{\partial k^*(\cdot)}{\partial s} > 0 \text{ and } \frac{\partial k^*(\cdot)}{\partial \delta} < 0 \\
\frac{\partial y^*(\cdot)}{\partial a} &> 0, \frac{\partial y^*(\cdot)}{\partial s} > 0 \text{ and } \frac{\partial y^*(\cdot)}{\partial \delta} < 0.
\end{aligned}$$

Equilibrium Without Population Growth and Technological Progress VIII

- **Proof of comparative static results:** follows immediately by writing

$$\frac{\tilde{f}\left(k^*\right)}{k^*} = \frac{\delta}{as},$$

which holds for an open set of values of k^*. Now apply the implicit function theorem to obtain the results.

- For example,

$$\frac{\partial k^*}{\partial s} = \frac{\delta\left(k^*\right)^2}{s^2 w^*} > 0$$

where $w^* = f\left(k^*\right) - k^* f'\left(k^*\right) > 0$.

- The other results follow similarly.

Equilibrium Without Population Growth and Technological Progress IX

- Same comparative statics with respect to a and δ immediately apply to c^* as well.
- But c^* will not be monotone in the saving rate (think, for example, of $s = 1$).
- In fact, there will exist a specific level of the saving rate, s_{gold}, referred to as the “golden rule” saving rate, which maximizes c^*.
- But cannot say whether the golden rule saving rate is “better” than some other saving rate.
- Write the steady-state relationship between c^* and s and suppress the other parameters:

$$\begin{aligned} c^*(s) &= (1-s) f(k^*(s)), \\ &= f(k^*(s)) - \delta k^*(s), \end{aligned}$$

- The second equality exploits that in steady state $sf(k) = \delta k$.

Equilibrium Without Population Growth and Technological Progress X

- Differentiating with respect to s,

$$\frac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - \delta\right]\frac{\partial k^*}{\partial s}. \tag{20}$$

- s_{gold} is such that $\partial c^*(s_{gold})/\partial s = 0$. The corresponding steady-state golden rule capital stock is defined as k^*_{gold}.

Proposition In the basic Solow growth model, the highest level of steady-state consumption is reached for s_{gold}, with the corresponding steady-state capital level k^*_{gold} such that

$$f'\left(k^*_{gold}\right) = \delta. \tag{21}$$

Golden Rule

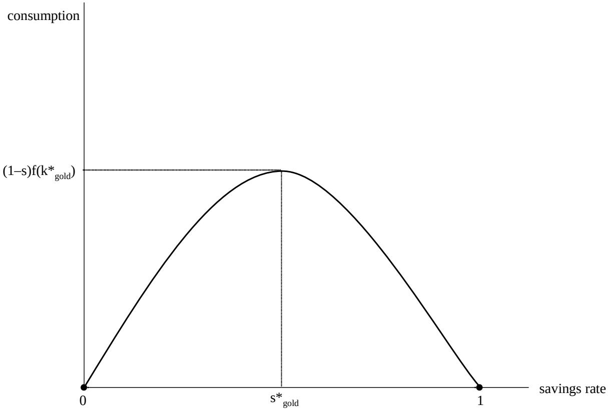


Figure 2.5: The “golden rule” level of savings rate, which maximizes steady-state consumption.

Proof of Proposition: Golden Rule

- By definition $\partial c^*(s_{gold})/\partial s = 0$.
- From Proposition above, $\partial k^*/\partial s > 0$, thus (20) can be equal to zero only when $f'(k^*(s_{gold})) = \delta$.
- Moreover, when $f'(k^*(s_{gold})) = \delta$, it can be verified that $\partial^2 c^*(s_{gold})/\partial s^2 < 0$, so $f'(k^*(s_{gold})) = \delta$ indeed corresponds a local maximum.
- That $f'(k^*(s_{gold})) = \delta$ also yields the global maximum is a consequence of the following observations:
  - $\forall\, s \in [0, 1]$ we have $\partial k^*/\partial s > 0$ and moreover, when $s < s_{gold}$, $f'(k^*(s)) - \delta > 0$ by the concavity of f, so $\partial c^*(s)/\partial s > 0$ for all $s < s_{gold}$.
  - by the converse argument, $\partial c^*(s)/\partial s < 0$ for all $s > s_{gold}$.
  - Therefore, only s_{gold} satisfies $f'(k^*(s)) = \delta$ and gives the unique global maximum of consumption per capita.

Equilibrium Without Population Growth and Technological Progress XI

- When the economy is below k^*_{gold}, higher saving will increase consumption; when it is above k^*_{gold}, steady-state consumption can be increased by saving less.
- In the latter case, capital-labor ratio is too high so that individuals are investing too much and not consuming enough (*dynamic inefficiency)*.
- But no utility function, so statements about “inefficiency” have to be considered with caution.
- Such dynamic inefficiency will not arise once we endogenize consumption-saving decisions.

Review of the Discrete-Time Solow Model

- Per capita capital stock evolves according to

$$k(t+1) = sf(k(t)) + (1-\delta)\,k(t)\,. \tag{22}$$

- The steady-state value of the capital-labor ratio k^* is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}. \tag{23}$$

- Consumption is given by

$$C(t) = (1-s)\,Y(t) \tag{24}$$

- And factor prices are given by

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t)\,f'(k(t)) > 0. \end{aligned} \tag{25}$$

Steady-State Equilibrium

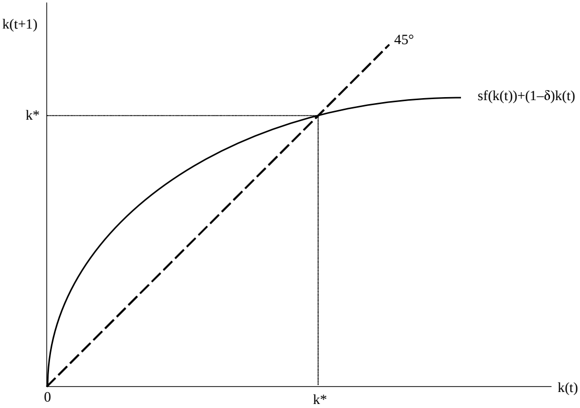


Figure 3.1: Steady-state capital-labor ratio in the Solow model.

Transitional Dynamics

- *Equilibrium path*: not simply steady state, but entire path of capital stock, output, consumption and factor prices.
  - In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus *the steady-state equilibrium*.
  - In economics, non-steady-state behavior also governed by optimizing behavior of households and firms and market clearing.
- Need to study the “transitional dynamics” of the equilibrium difference equation (22) starting from an arbitrary initial capital-labor ratio $k(0) > 0$.
- Key question: whether economy will tend to steady state and how it will behave along the transition path.

Transitional Dynamics: Review I

- Consider the nonlinear system of autonomous difference equations,

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t)), \qquad (26)$$

- $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$.
- Let $\mathbf{x}^*$ be a fixed point of the mapping $\mathbf{G}(\cdot)$, i.e.,

$$\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*).$$

- $\mathbf{x}^*$ is sometimes referred to as "an equilibrium point" of (26).
- We will refer to $\mathbf{x}^*$ as a stationary point or a *steady state* of (26).

Definition A steady state $\mathbf{x}^*$ is (locally) *asymptotically stable* if there exists an open set $B(\mathbf{x}^*) \ni \mathbf{x}^*$ such that for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ to (26) with $\mathbf{x}(0) \in B(\mathbf{x}^*)$, we have $\mathbf{x}(t) \to \mathbf{x}^*$. Moreover, $\mathbf{x}^*$ is *globally asymptotically stable* if for all $\mathbf{x}(0) \in \mathbb{R}^n$, for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$, we have $\mathbf{x}(t) \to \mathbf{x}^*$.

Transitional Dynamics: Review II

Simple Result About Stability

- Let $x(t), a, b \in \mathbb{R}$, then the unique steady state of the linear difference equation $x(t+1) = ax(t) + b$ is globally asymptotically stable (in the sense that $x(t) \to x^* = b/(1-a)$) if $|a| < 1$.
- Suppose that $g : \mathbb{R} \to \mathbb{R}$ is differentiable at the steady state x^*, defined by $g(x^*) = x^*$. Then, the steady state of the nonlinear difference equation $x(t+1) = g(x(t))$, x^*, is locally asymptotically stable if $|g'(x^*)| < 1$. Moreover, if $|g'(x)| < 1$ for all $x \in \mathbb{R}$, then x^* is globally asymptotically stable.

Transitional Dynamics in the Discrete Time Solow Model

Proposition Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (22) is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t)$ monotonically converges to k^*.

Proof of Proposition: Transitional Dynamics I

- Let $g(k) \equiv sf(k) + (1-\delta)k$. First observe that $g'(k)$ exists and is always strictly positive, i.e., $g'(k) > 0$ for all k.
- Next, from (22),
$$k(t+1) = g(k(t)), \tag{27}$$
with a unique steady state at k^*.
- From (23), the steady-state capital k^* satisfies $\delta k^* = sf(k^*)$, or
$$k^* = g(k^*). \tag{28}$$
- Recall that $f(\cdot)$ is concave and differentiable from Assumption 1 and satisfies $f(0) \geq 0$ from Assumption 2.

Proof of Proposition: Transitional Dynamics II

- For any strictly concave differentiable function,

$$f\left(k\right) > f\left(0\right) + kf'\left(k\right) \geq kf'\left(k\right), \tag{29}$$

- The second inequality uses the fact that $f\left(0\right) \geq 0$.
- Since (29) together with (23) implies that $\delta = sf\left(k^*\right)/k^* > sf'\left(k^*\right)$, we have $g'\left(k^*\right) = sf'\left(k^*\right) + 1 - \delta < 1$. Therefore,

$$g'\left(k^*\right) \in \left(0, 1\right).$$

- The Simple Result then establishes local asymptotic stability.

Proof of Proposition: Transitional Dynamics III

- To prove global stability, note that for all $k\left(t\right) \in \left(0, k^*\right)$,

$$\begin{aligned} k\left(t+1\right) - k^* &= g\left(k\left(t\right)\right) - g\left(k^*\right) \\ &= -\int_{k\left(t\right)}^{k^*} g'\left(k\right) dk, \\ &< 0 \end{aligned}$$

- First line follows by subtracting (28) from (27), second line uses the fundamental theorem of calculus, and third line follows from the observation that $g'\left(k\right) > 0$ for all k.

Proof of Proposition: Transitional Dynamics IV

- Next, (22) also implies

$$\begin{aligned} \frac{k(t+1)-k(t)}{k(t)} &= s\frac{f(k(t))}{k(t)}-\delta \\ &> s\frac{f(k^*)}{k^*}-\delta \\ &= 0, \end{aligned}$$

- Second line uses the fact that $f(k)/k$ is decreasing in k (from (29) above) and last line uses the definition of k^*.
- These two arguments together establish that for all $k(t) \in (0, k^*)$, $k(t+1) \in (k(t), k^*)$.
- An identical argument implies that for all $k(t) > k^*$, $k(t+1) \in (k^*, k(t))$.
- Therefore, $\{k(t)\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable.

Transitional Dynamics in the Discrete Time Solow Model III

- Stability result can be seen diagrammatically in the Figure:
  - Starting from initial capital stock $k(0) < k^*$, the economy grows towards k^*; *capital deepening* and growth of per capita income.
  - If the economy were to start with $k'(0) > k^*$, it would reach the steady state by decumulating capital and by contracting.

Proposition Suppose that Assumptions 1 and 2 hold, and $k(0) < k^*$, then $\{w(t)\}_{t=0}^{\infty}$ is an increasing sequence and $\{R(t)\}_{t=0}^{\infty}$ is a decreasing sequence. If $k(0) > k^*$, the opposite results apply.

- Thus far the Solow growth model has a number of nice properties, but no growth, except when the economy starts with $k(0) < k^*$.

Transitional Dynamics in Figure

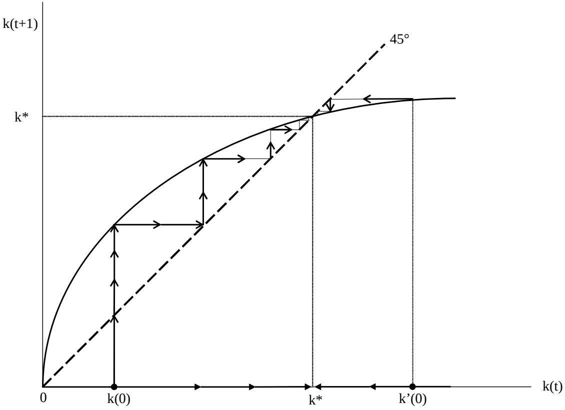


Figure 3.2: Transitional dynamics in the basic Solow model.